Pergamon

J. Appl. Maths Mechs, Vol. 60, No. 2, pp. 183–190, 1996

0021–8928/96 $24.00+0.00

S0021–8928(96)00024-X

THE SYNTHESIS OF OPTIMAL CONTROL IN A FOURTH-ORDER LINEAR SPEED OF RESPONSE PROBLEM†

Ye. R. BELOUSOVA and M. A. ZARKH

Ekaterinburg, Haifa

(*Received* 5 *May* 1995)

A well-known model of a moving oscillator is used to propose a new method for optimal feedback control synthesis, based on recalculation (correction) of an open-loop control along a real trajectory. The correction procedure involves solving a special differential equation for the switching times.

The subject of this paper is related to the investigations in [1–6].

1. STATEMENT OF THE PROBLEM

We shall investigate the problem of steering a control system

$$\dot{x}_1 = x_2, \ \dot{x}_2 = u, \ \dot{x}_3 = x_4, \ \dot{x}_4 = -x_3 + u, \ |u| \leqslant 1 \tag{1.1}$$

in the least possible time to an equilibrium position $x_i = 0$ ($i = 1, \ldots, 4$). In standard matrix notion, these equations may be written as $dx/dt = Ax(t) + Bu$. System (1.1) is a simplified model of a pendulum regulated by a force applied at the point of suspension. Other mechanical examples whose description reduces, under certain assumptions, to (1.1) may be found in [1–3].

System (1.1) has two pure imaginary roots and two non-zero roots. In addition, it satisfies the controllability condition. Consequently [7], the motion may be steered to zero from any starting position $x_0 \in \mathbf{R}^4$ in a finite time T. It is well-known [8] that an optimal open-loop control is of the on-off type (i.e. it takes values of ± 1 only) and is defined by a finite set of switching times t_i and the terminal time T. It is well-known [8] that an optimal open-loop control is of the on-off type (i.e. it takes values of ± 1 only) and is defined by a finite set of switching times t_i and the terminal time T. From the standpoint of applications, positional control (i.e. control based on feedback) is preferable to open-loop control, as it possesses the self-regulating property in cases where, in a practical system, factors extraneous to the model are active. Optimal positional control may be synthesized using a switching surface, which divides the space $\mathbf{R}^4$ into more parts. In one part the optimal control takes the value -1, and in the other, $+1$. In the problem just introduced, however, the switching surface is rather complicated and its construction and utilization require a powerful computer.

As a compromise, one might resort to a control scheme in which open-loop control is constantly corrected in real time. This kind of control would have the same properties as positional control, without requiring prohibitive computing expenses. The method proposed in this paper is applicable equally to the design of an optimal open-loop control at an arbitrary starting point and to real-time correction of such a control. Methods for correcting open-loop controls in problems with a linear performance functional and linear constraints on the terminal state were considered in [5, 6].

2. PARAMETRIZATION OF OPTIMAL PROCESSES

2.1. *The idea of parametrization.* In linear systems, Pontryagin's maximum principle may be interpreted as a way of parametrizing the family of extremal processes, that is, the processes that form the boundary of the accessibility region. The parameter is the boundary value of the solution of the adjoint system. The construction of an optimal control reduces, essentially, to determining a suitable parameter value. The set of extremal controls is independent of the starting point.

†*Prikl. Mat. Mekh.* Vol. 60, No. 2, pp. 189–197, 1996.

We will write the fundamental relation of the maximum principle in the form

$$l'\Phi(\tau)Bu(T-\tau) = \max_{|\xi|\le 1} l'\Phi(\tau)B\xi \tag{2.1}$$

where $\tau = T - t$ is reversed time, l is a vector with the geometrical meaning of the normal to the accessibility domain and $\Phi(\tau)$ is the Cauchy fundamental matrix. We shall use the well-known device of "pulling" trajectories out of the origin in reversed time

$$d\tilde{x} / d\tau = -(A\tilde{x}(\tau) + B\tilde{u}) \tag{2.2}$$

For any $T > 0$, every optimal trajectory of (2.2) is an optimal trajectory of (1.1) "pulled" out of the starting position $\tilde{x}(T)$. Indeed: $x(t) = \tilde{x}(\tau)$, $u(t) = \tilde{u}(\tau)$. Thus, given a starting state x^0, one must find a vector l determining the optimal trajectory of (2.2) that reaches x^0.

We put $p(\tau) = \Phi(\tau)B$ and replace (2.1) by the equation for the switching times (in reversed time). We obtain

$$l'p(\tau) = 0 \tag{2.3}$$

In the last transformation information concerning the sign of the control has been lost, since the roots of (2.3) determine a pair of sign-symmetric controls. We therefore introduce an additional parameter $\sigma = \tilde{u}(+0)$ which, together with the roots of Eq. (2.3), uniquely defines the control over the whole interval $[0, T]$. As a result it becomes possible to use an arbitrary non-zero representative of the straight line, λl, without worrying about the sign of the scalar product $l'p(\tau)$ of the roots.

For system (1.1) we have

$$\Phi(\tau) = \begin{Vmatrix} 1 & \tau & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\tau & \sin\tau \\ 0 & 0 & -\sin\tau & \cos\tau \end{Vmatrix}, \quad p(\tau) = \begin{Vmatrix} \tau \\ 1 \\ \sin\tau \\ \cos\tau \end{Vmatrix}$$

In scalar form Eq. (2.3) becomes

$$l_1\tau + l_2 + l_3 \sin\tau + l_4 \cos\tau = 0 \tag{2.4}$$

Usually, in order to fit the number of free parameters to the dimensions of the system, one imposes the additional restriction $|l| = 1$. In that case the three-parameter equivalent of (2.4) is the equality

$$a\tau + b = \sin(\tau + \varphi) \tag{2.5}$$

The solutions of Eq. (2.5) in the interval $[0, T]$ yield the switching times; they are the points at which the sine curve $\sin(\tau + \varphi)$ intersects the straight line $a\tau + b$. If the sine curve does not intersect the straight line anywhere in $[0, T]$, this means that the control applied up to time T has the constant value σ.

2.2. *Parametrization by switching times.* Let τ_i ($i = 1, \dots, n$) denote the ordered roots of Eq. (2.3) in the interval $[0, T]$. The following two propositions are stated without proof.

Proposition 1. Suppose $n \geqslant 3$. With the exception of the special case when the distance between any two roots is a multiple of 2π, one can find three distinct roots τ_{i_1}, τ_{i_2}, τ_{i_3} (e.g. two adjacent roots and any third one) such that the vectors $p(\tau_{i_1}), p(\tau_{i_2}), p(\tau_{i_3})$ are linearly independent. In the aforementioned special case the optimal control is constant.

Proposition 2. The trajectory segment $\tilde{x}(\tau)$, $0 \leqslant \tau \leqslant \tau_3$ lies on the switching surface.

Let us introduce a parametrization of optimal processes in which the parameters are the switching times. Suppose $n \geqslant 3$. Consider three roots of Eq. (2.3), denoting them by τ_{i_1}, τ_{i_2}, τ_{i_3}, such that the vectors $p(\tau_{i_1})$, $p(\tau_{i_2})$, $p(\tau_{i_3})$ are linearly independent. Then a non-zero representative of the straight lines λl may be expressed in terms of τ_{i_1}, τ_{i_2}, τ_{i_3}. Retaining the old notation l, we have

$$l = \sum_{k=1}^{4} \det(p(\tau_{i_1}),\ p(\tau_{i_2}),\ p(\tau_{i_3}),\ e_k)e_k$$

Here e_k ($k = 1, \ldots, 4$) is an orthonormal basis of the space $\mathbf{R}^4$ and det is the determinant of the matrix with columns $p(\tau_{i_1}), p(\tau_{i_2}), p(\tau_{i_3}), e_k$. Equation (2.3) becomes

$$g(\tau) \triangleq \det(p(\tau_{i_1}), p(\tau_{i_2}), p(\tau_{i_3}), p(\tau)) = 0 \tag{2.6}$$

The times $\tau = \tau_{i_1}, \tau_{i_2}, \tau_{i_3}$ are automatically roots of Eq. (2.6). The other switching times are implicit functions of $\tau_{i_1}, \tau_{i_2}, \tau_{i_3}$. However, the conditions of the Implicit Function theorem are not always satisfied. Accordingly, we introduce a classification of optimal processes.

We will call a root of Eq. (2.3) regular (or: a switching) if it is neither 0 nor T and has multiplicity 1, and irregular otherwise.

We will call an optimal process (OP) regular if $n \geqslant 3$ and all the roots of Eq. (2.3) are regular. An OP with $n \geqslant 3$ is irregular if at least one irregular root of Eq. (2.3) exists in the interval $[0, T]$. An OP for which Eq. (2.3) has less than three roots will also be considered irregular.

In the regular case, any three switchings may be taken as parameters. The equation for determining all switching times is (2.6).

The subset of irregular OPs admits of a special parametrization. We shall introduce certain relationships that define (for fixed T) *two-parameter* families of OPs with a single irregular root. These relationships are important in the context of numerical constructions. The other cases (double or triple irregularity) occur less probably in that context and will therefore not be described in this paper.

Suppose $n \geqslant 3$. Assume that there are at least two switchings and one irregular root. The existence of an irregular root imposes an additional restriction on the vector l. To determine vectors l corresponding to irregular processes, we will use the irregular root in the expression for l. Consequently, along with the general relationship (2.6), which includes all OPs without exception, we obtain special analogues of (2.6) for irregular OPs.

If $\tau = 0$ is a root, we take not any triple of vectors orthogonal to l but a triple including the vector $p(0)$ and the two vectors $p(\bar{\tau}), p(\bar{\bar{\tau}})$ with regular roots $\bar{\tau}, \bar{\bar{\tau}}$. Equation (2.6) becomes

$$\det(p(\bar{\tau}),\ p(\bar{\bar{\tau}}),\ p(0),\ p(\tau)) = 0 \tag{2.7}$$

Similarly, for the root $\tau = T$

$$\det(p(\bar{\tau}),\ p(\bar{\bar{\tau}}),\ p(T),\ p(\tau)) = 0 \tag{2.8}$$

If the irregularity is due to the existence of a multiple root τ_* (the sine curve is tangent to the straight line), the vector l is orthogonal to $p(\tau_*)$ and $\dot{p}(\tau_*)$. Taking this pair of vectors and the vector $p(\bar{\tau})$ with a regular root $\bar{\tau}$, we obtain

$$\det(p(\bar{\tau}),\ p(\tau_*),\ \dot{p}(\tau_*),\ p(\tau)) = 0 \tag{2.9}$$

Equations (2.7)–(2.9) yield a parametrization of irregular OPs with one singularity. Similar derivations yield relationships for irregular OPs characterized by the existence of two or more singularities.

Note that when $n \geqslant 3$ and the OP is regular or irregular in the sense just considered, the direction l is unique.

Let $n = 2$ and suppose that the roots τ_1 and τ_2 are regular To such an OP with two switchings there corresponds a two-dimensional cone of vectors l, for each of which Eq. (2.3) has the same pair of regular roots τ_1, τ_2. There are no other roots if l is in the interior of the cone. For the generators of the cone, a third root appears (though the structure of the OP is unchanged). One of the generators produces the irregular root $\max[0, \tau_{*1}]$, the other, the root $\min[T, \tau_{*2}]$, where τ_{*1} and τ_{*2} are the roots of the equation

$$\det(p(\tau_1),\ p(\tau_2),\ p(\tau_*),\ \dot{p}(\tau_*)) = 0$$

nearest τ_1 on the left and nearest τ_2 on the right. The combination $\tau_{*1} > 0, \tau_{*2} < T$ is impossible. Thus, each specific pair τ_1, τ_2 may be completed by a third root, in two ways. In other words, the OP in question is described by two of the relationships (2.7)–(2.9). In (2.7) and (2.8) the roots $\bar{\tau}, \bar{\bar{\tau}}$ are the times τ_1, and τ_2, while in (2.9) the roots $\bar{\tau}$ and τ_* are the times τ_1 and $\tau_{*1}(\tau_{*2})$.

We have thus successfully parametrized regular OPs and two-parameter irregular OPs in terms of switching times.

2.3. *Singular surfaces.* If one varies not only the parameters $\bar{\tau}, \bar{\bar{\tau}}$ but also T, then the ends $\tilde{x}(T)$ of the irregular trajectories defined by (2.7)–(2.9) form a system of three-dimensional singular surfaces that divides $\mathbf{R}^4$ into domains of regular OPs. In particular, a singular surface of type (2.8) is a switching surface. If an irregular OP is described by Eq. (2.8) and it has more than two switchings, then the corresponding trajectory pierces the switching surface at time T, crossing from one regular domain into another. In all other cases, irregular optimal trajectories move along singular surfaces.

The points of intersection of two three-dimensional singular surfaces of different types (in particular, trajectories with a single switching of the control) generate two-dimensional singular manifolds. The latter meet on singular curves. In particular, a singular curve is a pair of trajectories corresponding to the constant control $u = \pm 1$. We have described the structure of a three-dimensional switching surface and singular manifolds of lower dimensions on it elsewhere.†

Let us consider a three-dimensional singular surface as the boundary between two regular domains, and describe the different ways in which the number of switching times changes as the surface is crossed. This information will be needed in the next section. If the number of regular roots on the surface exceeds two, the irregular root becomes regular in one domain (in the case of (2.7), (2.8)) or generates a pair of closely situated regular roots (in the case of (2.9)). In the second domain the irregular root disappears. The situation in which the number of regular roots is two is characterized by the presence of two ("interchangeable") additional irregular roots (these are 0, T, or 0, τ_{*2}, or τ_{*1}, T). When the singular surface is crossed, one of the irregular roots disappears and the other becomes regular.

The proposed parametrization (2.6)–(2.9) may also be used for other control systems. It reduces the number of variables participating in the specification of the OP and is also a convenient tool to describe irregularities.

3. CORRECTION PROCEDURE

Let us consider a regular OP with switching times $\tau_1, \ldots, \tau_n$ in an interval $[0, T]$; let σ be the sign of the controls $\tilde{u}$ in the interval $(0, \tau_1)$. By the Cauchy formula

$$\tilde{x} = \tilde{x}(T) = \sigma \sum_{i=1}^{n+1} (-1)^i \int_{\tau_{i-1}}^{\tau_i} p(\tau - T)d\tau; \quad \tau_0 = 0, \quad \tau_{n+1} = T \tag{3.1}$$

Equations (2.6) and (3.1) enable us to determine the variations of the end $\tilde{x}(T)$ of the trajectory and the switching times as the parameters $\tau_1, \tau_2, \tau_3, T$ are varied

$$d\tilde{x} = MCds, \quad dS = CdS \tag{3.2}$$

$$s = (\tau_1, \tau_2, \tau_3, T)', \quad S = (\tau_1, \ldots, \tau_n, T)'$$

$$M = \left\| \frac{\partial \tilde{x}_k}{\partial \tau_j} \right\|_{k=1,\ldots,4}^{j=1,\ldots,n+1}, \quad C = \left\| \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \multicolumn{3}{c}{\left\| \dfrac{\partial \tau_i}{\partial \tau_k} \right\|_{i=4,\ldots,n}^{k=1,2,3}} & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \right\|$$

$$\partial \tilde{x} / \partial T = -(A\tilde{x} + B\tilde{u}(T)), \quad \tilde{u}(T) = (-1)^n \sigma$$

$$\partial \tilde{x} / \partial \tau_j = (-1)^j 2\sigma p(\tau_j - T), \quad j = 1, \ldots, n$$

†BELOUSOVA Ye. R. and ZARKH M. A., The switching surface in a linear fourth-order speed-of-response problem. Ekaterinburg, 1992. Deposited at VINITI 24.07.92, No. 2442–B92.

$$\partial\tau_i / \partial\tau_1 = -\det(\dot{p}(\tau_1),\ p(\tau_2),\ p(\tau_3),\ p(\tau_i)) / d$$
$$\partial\tau_i / \partial\tau_2 = -\det(p(\tau_1),\ \dot{p}(\tau_2),\ p(\tau_3),\ p(\tau_i)) / d$$
$$\partial\tau_i / \partial\tau_3 = -\det(p(\tau_1),\ p(\tau_2),\ \dot{p}(\tau_3),\ p(\tau_i)) / d$$
$$d = \det(p(\tau_1),\ p(\tau_2),\ p(\tau_3),\ \dot{p}(\tau_i))$$

For a regular process, the matrix MC is non-singular, and it therefore follows from (3.2) that

$$dS = C(MC)^{-1} d\tilde{x} \tag{3.3}$$

We can now go on to construct an optimal open-loop control (OC) for an arbitrarily specified starting point $x_0 \in \mathbf{R}^4$. Suppose we have a certain open-loop control (not related in any way to x_0) in a certain time interval $[0, T_{00}]$, given by a vector S_{00} and a parameter σ_{00}. The first n components of S_{00} must be roots of Eq. (2.6), the others need not even be in the interval $[0, T_{00}]$. Let $x_{00} = \tilde{x}(T_{00})$ be the initial state, evaluated by formula (3.1) and corresponding to this control. We require the variation of the end of the trajectory to be directed toward the point x_0. To that end we express $d\tilde{x}$ in the form

$$d\tilde{x} = (x_0 - x_{00}) d\mu \tag{3.4}$$

where μ is a scalar parametrizing the interval $[x_{00}, x_0]$: $x(\mu) = x_{00} + \mu(x_0 - x_{00})$, $0 \leqslant \mu \leqslant 1$, $x(0) = x_{00}$, $x(1) = x_0$. Substituting (3.4) into (3.3), we obtain the equation

$$dS / d\mu = C(MC)^{-1}(x_0 - x_{00}) \tag{3.5}$$

If the interval $[x_{00}, x_0]$ does not cut singular surfaces, then, integrating (3.5) with initial condition $S(0) = S_{00}$, we find an OC for the point x_0, given by a pair $S_0 = S(1)$, $\sigma_0 = \sigma_{00}$. Integration of Eq. (3.5) requires recalculation of the matrices C and M at each step, using explicit formulae.

Of course, it is no known a priori whether the segment $[x_{00}, x_0]$ cuts singular surfaces. Therefore, when integrating (3.5), one must check for approach to a singular surface and, on crossing it, transform the vector S in accordance with the scheme described in Section 2.3. In cases when the switching time drifts or vanishes, the sign of σ is reversed.

Remark. With the exception of specially selected cases, the probability that the segment $[x_{00}, x_0]$ will cut a singular manifold of dimension less than three is zero. That is why we described only three-dimensional singular surfaces in Section 2.3.

To demonstrate the performance of the algorithm, consider the construction of an OC for the initial state $x_0 = (0.05, 0.50, 6.00, -2.00)'$. The initial control was the pair $S_{00} = (2.00, 5.00, 7.50, 8.00)'$, $\sigma_{00} = +1$. The corresponding starting point is $x_{00} = (4.75, -1.00, 4.80, -0.87)'$, calculated by formula (3.1). The control S_{00}, σ_{00} is optimal for x_{00}, because only the times 2, 5 and 7.5 are roots of Eq. (2.6) in the interval [0. 8]. Figure 1 is a graph showing the variation of the components of the vector S along the interval $[x_{00}, x_0]$. The interval $[x_{00}, x_0]$ cuts two singular surfaces, and crossing either of them entailed addition of a new switching. Arrival at the surface was determined by the equality $g(0) = 0$, while the inequality $\tau_1 > 0$ indicated that the zero root must be included in the vector S. The algorithm finally produced an OC for the starting position x_0, defined by the pair $S_0 = (0.57, 2.53, 6.06, 9.82, 11.52, 12.08)'$, $\sigma_0 = +1$. The time required for optimum steering from x_0 to zero was 12.08.

As a second illustration, consider the problem of constructing an OC for initial points on the straight line $X_1 = \{(x_1{:}\ 0{:}\ 0{:}\ 0)'{:}\ x_1 \in \mathbf{R}^1\}$. It has been shown [1] that if $x_1 \neq \pm 4\pi^2 k^2$, $k \in N$, the OC has three switchings such that

$$\tau_2 = T/2,\ \tau_3 = T - \tau_1 \tag{3.6}$$

If $x_1 = \pm 4\pi^2 k^2$, the OC has one switching $\tau_1 = 2\pi k$, and the steering time is $T = 4\pi k$. The previously proposed method [1] for determining the times τ_1, τ_2, τ_3, T in the general case reduces to solving a certain transcendental equation in T.

Following the approach proposed in this paper, we introduce a differential equation describing the variation of the OP as the starting point "moves" along the straight line X_1. The solution of this equation yields the OC as a function of x_1.

For processes with three switchings, C is the identity matrix, so that the relationship $d\tilde{x} = MCds$ of (3.2) with $\sigma = -1$ becomes

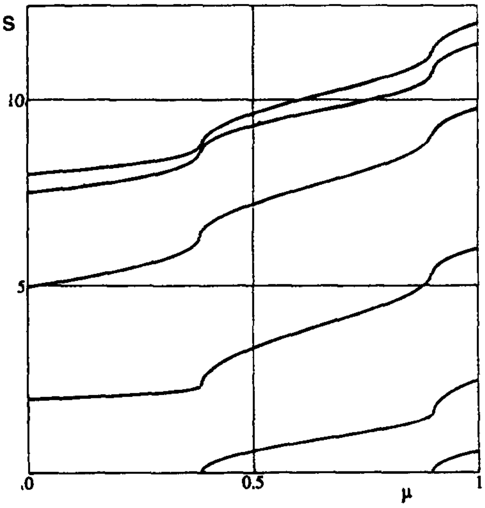

Fig. 1.

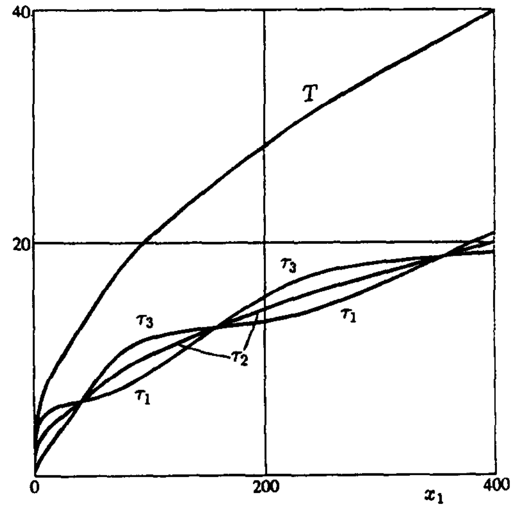

Fig. 2.

$$2\begin{Vmatrix} -(\tau_1-T) & (\tau_2-T) & -(\tau_3-T) & 0 \\ -1 & 1 & -1 & 1/2 \\ -\sin(\tau_1-T) & \sin(\tau_2-T) & -\sin(\tau_3-T) & 0 \\ -\cos(\tau_1-T) & \cos(\tau_2-T) & -\cos(\tau_3-T) & 1/2 \end{Vmatrix} \begin{Vmatrix} d\tau_1 \\ d\tau_2 \\ d\tau_3 \\ dT \end{Vmatrix} = \begin{Vmatrix} dx_1 \\ dx_2 \\ dx_3 \\ dx_4 \end{Vmatrix} \tag{3.7}$$

Let us consider variations of the form $dx = (dx_1; 0; 0; 0)'$ and substitute expressions (3.6) for τ_2 and τ_3 and the analogous expressions for the differentials $d\tau_2$ and $d\tau_3$ into (3.7). It will suffice to use just one row of the matrix equality (3.7). Selecting the third row and solving it for $d\tau_1/dT$, we obtain

$$d\tau_1 / dT = (\sin\tau_1 - \sin(T/2)) / [2(\sin(\tau_1 - T) + \sin\tau_1)] \tag{3.8}$$

Using formula (3.1), it is not difficult to check that the OC $S_{00} = (\pi/2; \pi; 3\pi/2; 2\pi)'$, $\sigma_{00} = -1$ corresponds to the starting position $x_{00} = (\pi^2/2; 0; 0; 0)'$. Thus, a suitable initial condition for (3.8) is $\tau_1(2\pi) = \pi/2$. Integrating (3.8), we obtain $\tau_1 = \tau_1(T)$. Using (3.6), we get $\tau_2 = \tau_2(T)$, $\tau_3 = \tau_3(T)$. Equation (3.1) gives $x_1 = x_1(T)$. Hence one can express T in terms of x_1. Figure 2 plots the functions τ_1, τ_2, τ_3, T against x_1.

Thus Eq. (3.8) enables one to find OCs immediately for all points of the straight line X_1.

4. CORRECTING FEEDBACK

We shall now consider system (1.1) as an idealized, standard model, used to design controls in a real system whose dynamics may depart from that of (1.1) (e.g. owing to noise). Let $x(t)$ denote the position of the standard system at time t and $\bar{x}(t)$ the position of the real system. We shall assume that the vectors $x(t)$ and $\bar{x}(t)$ are of the same length. Suppose that both systems start from the same position x^0 at time $t_0 = 0$. The times at which the real system will be corrected will depend on a special parameter $\varepsilon > 0$. At these times the position of the standard system will change abruptly.

We obtain an OC $S(x^0)$, $\sigma(x^0)$ and, thereby, the standard motion $x(\cdot)$ of system (1.1) from the point x^0. The standard open-loop control governs the real system up to the first time t_1^* at which $|x(t_1^*) - \bar{x}(t_1^*)| \geqslant \varepsilon$. An OC for the position $x(t_1^*)$ of the standard system is known: $S(x(t_1^*)) = S(x^0)|_{[0,\, T(x^0)-t_1^*]}$, $\sigma(x(t_1^*)) = \sigma(x^0)$. After carrying out the correction procedure, with the roles of x_{00}, x_0 in (3.5) taken by the points $x(t_1^*)$ and $\bar{x}(t_1^*)$, respectively, one finds an OC for the state $\bar{x}(t_1^*)$. This yields a new standard motion from the point $\bar{x}(t_1^*)$. The same OC continues to govern the system until the next correction time t_2^*, when $|x(t_2^*) - \bar{x}(t_2^*)| \geqslant \varepsilon$, and so on.

The correction times may also be selected differently, e.g. they may be made to depend on a given time step Δ. As $\Delta \to 0$ one obtains a continuous correction regime.

Remarks. 1. In case less than three switching times remain in the vector $S(x(t_i^*))$ (i.e. the point $x(t_i^*)$ lies on the switching surface), one introduces additional switching times in order to ensure that the input to the procedure should contain corrections of a regular OP for some point close to $x(t_i^*)$.

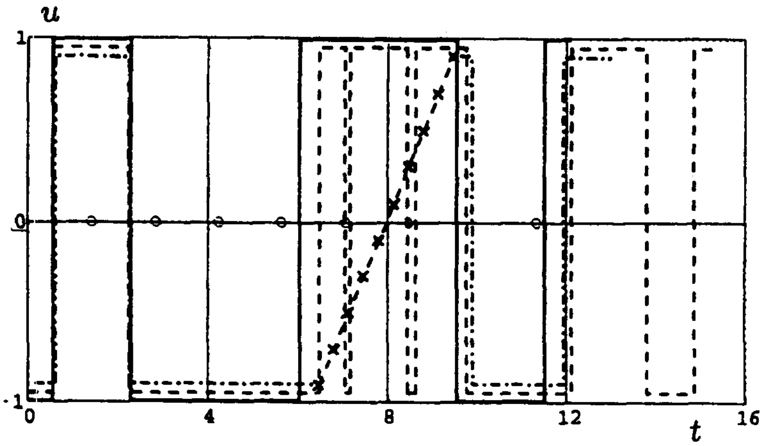

Fig. 3.

If only one switching τ_1 remains in the vector $S(x(t_i^*))$, additional switchings are introduced at $\tau_1 - \delta$, $\tau_1 + \delta$ (where δ is a small positive number). If the vector $S(x(t_i^*))$ contains two switchings τ_1 and τ_2, one introduces either two additional switchings $\tau_* - \delta$, $\tau_* + \delta$ ($\tau_* \in (0, \tau_1)$), or one additional switching δ (in the latter case the parameter σ changes sign). The time τ_* and the appropriate version of those described are determined by the rule given in Section 2.2. If there are no switchings in $\tau_* - \delta$, $\tau_* + \delta$ ($\tau_* \in (0, \tau_1)$) one considers switchings $T - 3\delta$, $T - 2\delta$, $T - \delta$.

2. Assuming that the real motion is described by a differential equation with a known or accurately measured right-hand side f, one can write down the continuous correction equation

$$dS/dt = C(MC)^{-1}(f - (A\bar{x} + Bu)) \tag{4.1}$$

Equation (4.1) has a classical solution in regular domains, which changes form after passing through singular surfaces of types (2.7) and (2.9). Solutions on switching surfaces are defined as in Filippov [9].

For correction in a regular domain with a small time step Δ, we deduce from (4.1) that

$$S(t+\Delta) = S(t) + C(MC)^{-1}(\bar{x}(t+\Delta) - x(t+\Delta)) \tag{4.2}$$

Thus, we have a formula that is identical with the single-step solution of Eq. (3.5). To carry out calculations using this formula one uses the states $\bar{x}$ and x at discrete times.

If the interval $[x(t + \Delta), \bar{x}(t + \Delta)]$ cuts a singular surface at some point $x(t + \Delta) + \mu(\bar{x}(t + \Delta) - x(t + \Delta))$, $0 \leq \mu \leq 1$, Eq. (4.2) is replaced by two equations

$$S_* = S(t) + \mu C(MC)^{-1}(\bar{x}(t+\Delta) - x(t+\Delta))$$

$$S(t+\Delta) = S^* + (1-\mu)C(MC)^{-1}(\bar{x}(t+\Delta) - x(t+\Delta))$$

where the vector S_* is transformed into S^* in accordance with the crossed singularity.

Let us demonstrate the performance of the correcting feedback for a "real" system described by the equation

Table 1

t_i^*	$S(x(t_i^*))$	$S(\bar{x}(t_i^*))$
1.41	(0.56; 2.53; 6.06; 9.82; 10.67)	(0.70; 2.71; 6.17; 10.06; 10.88)
2.84	(0.70; 2.71; 6.17; 9.46)	(0.84; 2.88; 6.21; 9.63)
4.24	(0.84; 2.88; 6.21; 8.23)	(0.93; 2.98; 6.20; 8.33)
5.64	(0.93; 2.98; 6.20; 6.93)	(0.99; 3.05; 6.18; 7.00)
7.06	(0.99; 3.05; 5.58)	(1.07; 3.21; 5.67; 5.76)
8.45	(1.07; 3.21; 4.37)	(1.14; 3.32; 4.47; 4.63)
11.32	(1.14; 1.77)	(0.56; 1.62; 3.33; 4.11)

$$\frac{d\bar{x}}{dt} = A\bar{x}(t) + Bu + v(t), \quad v(t) = \begin{cases} 0,\ 1(1,\ 1,\ \sin t,\ \cos t)' & t \in [0, 3\pi] \\ \mathbf{0}, & t > 3\pi \end{cases}$$

where the matrices A and B are the same as in (1.1), and $v(t)$ is interpreted as some interference.

Consider the starting point $x^0 = (0.05, 0.50, 6.00, -2.00)'$. The corresponding standard motion was found in Section 3. Figure 3 shows the standard optimal open-loop control for x^0 (the solid line), the real control synthesized by correcting feedback (the dashed line; $\varepsilon = 0.3$; the correction times are indicated by small circles) and the actual control obtained by continuous correction (the dash–dot line; the correction times are selected with a step $\Delta = 0.05$; the inclined dashed line indicated by crosses corresponds to a "chattering" control). Table 1, drawn up for the case $\varepsilon = 0.3$, shows the correction times and the input and corrected switching vectors. The last correction was carried out after the interference had been cut off. All motions arrive at the origin.

We wish to thank V. S. Patsko for useful discussions and comments on this paper.

REFERENCES

1. CHERNOUS'KO F. L., AKULENKO L. D. and SOKOLOV B. N., *Control of Vibrations*. Nauka, Moscow, 1980.
2. DOBRYNINA I. S. and CHERNOUS'KO F. L., Bounded control of a linear fourth-order system. *Izd. Ross. Akad. Nauk, Tekhn. Kibern.* 6, 94–100, 1992.
3. CHERNOUS'KO, F. L., The construction of a bounded control in oscillatory systems. *Prikl. Mat. Mekh.* **52**, 4, 549–558, 1988.
4. BELOUSOVA Ye. R. and ZARKH M. A., Construction of a switching surface in a linear fourth-order time-optimum problem. *Izd. Ross. Akad. Nauk, Tekh. Kibern.* 6, 126–139, 1994.
5. GABASOV R., KIRILLOVA F. M. and KOSTYUKOVA O. I., A method for the optimal control of the motion of a dynamical system under constantly acting perturbations. *Prikl. Mat. Mekh.* **56**, 5, 854–863, 1992.
6. GABASOV R., KIRILLOVA F. M. and KOSTYUKOVA O. I., Optimization of a linear control system in real time. *Izd. Ross. Akad. Nauk, Tekh. Kibern.* 4, 3–19, 1992.
7. LEE E. B. and MARKUS L., *Foundations of Optimal Control Theory*. John Wiley, New York, 1967.
8. PONTRYAGIN L. S., BOLTYANSKII V. G., GAMKRELIDZE R. V. and MISHCHENKO Ye. F., *Mathematical Theory of Optimal Processes*. Fizmatgiz, Moscow, 1976.
9. FILIPPOV A. F., *Differential Equations with Discontinuous Right-hand Side*. Nauka, Moscow, 1985.

Translated by D.L.